Thierry Vialar

Complex and Chaotic Nonlinear Dynamics

Advances in Economics and Finance, Mathematics and Statistics

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