1 Basic Reminders about Production Function

1.1 Production Function With Complementary Factors

$$Y = min\left(\frac{K}{a}, \frac{L}{b}\right) \tag{1}$$

(isoquant $Y = 1$).

1.2 Production Function With Substituable Factors

$Y = F(K, L)$, F continuous and twice derivable, $F'_K > 0$, $F'_L > 0$,

$$\frac{\partial^2 F}{\partial K^2} \leq 0 \text{ and } \frac{\partial^2 F}{\partial L^2} \leq 0 \tag{2}$$

and the Inada conditions:

$$\lim_{K\to\infty} F'_K = \lim_{L\to\infty} F'_L = 0 \text{ and } \lim_{K\to 0} F'_K = \lim_{L\to 0} F'_L = +\infty. \tag{3}$$

1.3 Marginal Rate of Substitution

The differential of the production function $Y = F(K, L)$:

$$dY = F'_K dK + F'_L dL \tag{4}$$

along an isoquant $dY = 0$ and the *Marginal Rate of Substitution* (MRS) of *Capital* to *Labor* can be defined by

$$MRS_{KL} = -\frac{dK}{dL} = \frac{F'_L}{F'_K} \tag{5}$$

1.4 Elasticity of substitution

Elasticity of substitution of capital to labor:

$$\sigma_{KL} = \frac{d\ln(K/L)}{d\ln(MRS_{KL})} = -\frac{d\ln(K/L)}{d\ln(F'_L/F'_K)} \tag{6}$$

Hypothesis of remuneration of factors to their marginal productivity,
u nominal cost of capital,
w nominal rate of wage,
p level of prices

$$\begin{cases} F'_K = \dfrac{u}{p} \\ F'_L = \dfrac{w}{p} \end{cases} \tag{7}$$

$$MRS_{KL} = \frac{w}{p} \tag{8}$$

$$\sigma_{KL} = \frac{d\ln\left(\dfrac{K}{L}\right)}{d\ln\left(\dfrac{w}{u}\right)} = -\frac{d\ln\left(\dfrac{K}{L}\right)}{d\ln\left(\dfrac{u}{w}\right)} \tag{9}$$

1.5 Returns to Scale

. Constant if $\forall\lambda$, $F(\lambda K, \lambda L) = \lambda F(K, L)$
. Decreasing if $\forall\lambda > 1$, $F(\lambda K, \lambda L) > \lambda F(K, L)$
. Increasing if $\forall\lambda > 1$, $F(\lambda K, \lambda L) < \lambda F(K, L)$
Constant Returns to Scale
Euler theorem:

$$Y = F'_K K + F'_L L \tag{10}$$

that means

$$Y = \frac{u}{p}K + \frac{w}{p}L \Leftrightarrow pY = \mathrm{u}K + wL \tag{11}$$

Product Exhaustion Theorem
Zero-Profit
Reasoning per capita:
$y = Y/L$ per capita product (or mean productivity of labor)
$k = K/L$ per capita capital
$y = F(K/L, 1) = F(k, 1) = f(k)$

$$\begin{cases} F'_K = f'(k) \\ F'_L = f(k) - kf'(k) \end{cases} \tag{12}$$

$$\begin{cases} \dfrac{u}{p} = f'(k) \\ \dfrac{w}{p} = f(k) - kf'(k) \end{cases} \tag{13}$$

Frontier of the prices of factors

$$\frac{w}{p} = f\left[f'^{-1}\left(\frac{u}{p}\right)\right] - \frac{u}{p}f'^{-1}\left(\frac{u}{p}\right) = \frac{w}{p}\left(\frac{u}{p}\right) \tag{14}$$

1.6 Cobb-Douglas Function

$$Y = AK^{\alpha}L^{\beta} \tag{15}$$

with $0 < \alpha$, $\beta < 1$

$$\begin{cases} MRS_{KL} = \dfrac{\beta}{\alpha}\dfrac{K}{L} \\ \sigma_{KL} = 1 \end{cases} \tag{16}$$

$$F'_L = \beta\frac{Y}{L} \text{ and } F'_K = \alpha\frac{Y}{K} \tag{17}$$

$$\frac{wL}{pY} = \beta \text{ and } \frac{\text{uK}}{pY} = \alpha \tag{18}$$

If $\beta = 1 - \alpha$, constant returns to scale. Then

$$y = Ak^{\alpha} \tag{19}$$

and frontier of the prices of factors:

$$p = \frac{1}{A\alpha^{\alpha}(1-\alpha)^{1-\alpha}} u^{\alpha} w^{1-\alpha} \tag{20}$$

1.7 C.E.S Function

Function of Constant Elasticity of Substitution (CES)

$$Y = \left[aK^{-\gamma} + (1-a)L^{-\gamma}\right]^{-\mu} \tag{21}$$

with $0 < a < 1$, $\gamma > -1$, $\gamma \neq 0$

$$\begin{cases} MRS_{KL} = \dfrac{1-a}{a}\left(\dfrac{K}{L}\right)^{1+\gamma} \\ \sigma_{KL} = \dfrac{1}{1+\gamma} \end{cases} \tag{22}$$

Constant returns to scale if $\gamma\mu = 1$, increasing if $\gamma\mu > 1$, decreasing if $\gamma\mu < 1$. $\gamma\mu$ is the *scale elasticity*. The per capita formulation of the CES function with constant returns to scale:

$$y = \left[ak^{-\gamma} + (1-a)\right]^{-1/\gamma}$$