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# Complex and Chaotic Nonlinear Dynamics

Advances in Economics and Finance, Mathematics and Statistics



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